

Entropy and black hole horizons

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Abstract

The standard approach of counting the number of eigenmodes of N scalar fields near the horizon is used as a basis to provide a simple statistical mechanical derivation of the black hole entropy in two and four dimensions. The Bekenstein formula $S = \frac{A}{4G\hbar}$ and the two-dimensional entropy $S = 2M/\lambda\hbar$ are naturally obtained (up to a numerical constant of order 1). This approach provides an explanation on why the black hole entropy is of order $1/\hbar$ and why it is independent of the number of field-theoretical degrees of freedom.

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1. Horizons of evaporating black holes

An important problem in quantum gravity is understanding the origin of Bekenstein entropy of black holes,

$$S = \frac{A}{4G\hbar} , \quad A = 4\pi r_s^2 = 16\pi G^2 M^2 . \quad (1.1)$$

Bekenstein suggested that S , more than a mere analogy to the thermodynamical entropy, should dictate the number of possible quantum states of a black hole [1]. This interpretation was put on a firmer ground after Hawking's discovery of black hole radiation [2]. Subsequently it was argued that a statistical mechanical origin of the Bekenstein entropy requires an enormous reduction of the number of gauge-invariant degrees of freedom in quantum gravity, to the extent that one space dimension might be superfluous [3,4]. Recently different statistical mechanical derivations of the Bekenstein entropy, as a geometric or entanglement entropy, have been investigated (see e.g. refs. [5,6,7,8]). The present derivation is closer in spirit to the "brick wall" model of ref. [5], but in addition we shall take into account the black hole evaporation. Although an infinite result was found in [5,6], there the black hole was considered as a fixed background. Here it will be shown that the result is finite upon incorporating the back reaction on the geometry due to Hawking radiation. The standard entropy formula of two-dimensional dilaton gravity will emerge (up to a numerical factor of order 1). While the inclusion of back-reaction effects will lead to a finite result in four-dimensional gravity as well, the resulting quantum-field-theory entanglement entropy is much larger than the Bekenstein entropy. We will argue that the correct Bekenstein entropy naturally follows from the assumption that the quantum mechanical information is encoded in the Hawking radiation.

Classically, in the Schwarzschild frame of an external observer, infalling matter reaches the event horizon in an infinite time. At quantum level, assuming that at some moment the black hole completely disappears, matter crosses the event horizon at some calculable, finite Schwarzschild time (this time can be explicitly obtained, e.g. in the model of ref. [9]). Infinite Schwarzschild time now corresponds to another horizon, which is just beyond the event horizon, and we shall call it (following ref. [10]) the *ultimate* horizon (see also refs. [2,9]). In the classical theory the event horizon and the ultimate horizon coincide. When the evaporation of the black hole is taken into account, quantities which diverged on the classical black hole geometry (due to the presence of the event horizon), such as redshifts, are now finite. In particular, the integral of the outgoing energy density flux is no longer infinity, since the Hawking flux measured by any time-like outside observer stops after he enters into the vacuum region (in fact this integral reproduces, as expected, the total ADM energy of original collapsing matter). A natural question is whether also the entropy associated with quantum fields near the horizon is finite when the black hole evaporation is incorporated. This turns out to be the case. However, in four dimensions the resulting field theoretic entropy will still be far larger than the Bekenstein entropy, and a cut-off will be needed.

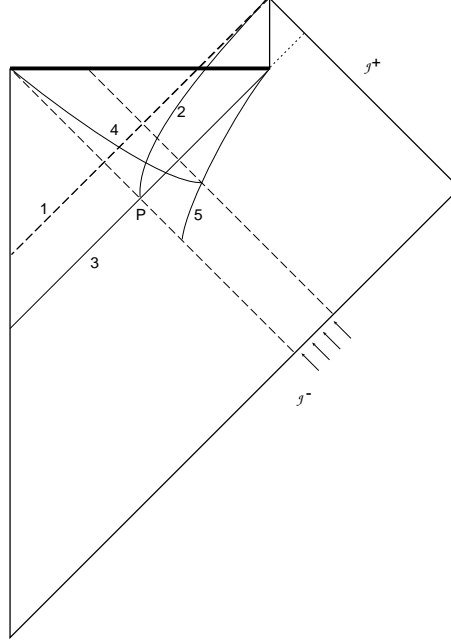


Fig. 1: Penrose diagram for the process of formation and evaporation of a black hole.

Consider the process of black hole formation and evaporation, and let us assume that the final geometry is given by the Minkowski metric with radial coordinate r_* and time t . We will refer to different horizons (see fig. 1):

- 1) *Ultimate horizon*: The null line passing through the boundary point of the future null infinity \mathcal{J}_+ . Here the Minkowski time t of the outside observers goes to infinity.
- 2) *Hidden horizon*: To construct it, a null line is drawn from \mathcal{J}_- to the point where the boundary becomes space-light. This line intersects the event horizon at some point P . Let r_{*h} be the value of r_* at this point. The *hidden horizon* is the time-like surface of space-time points with $r_* = r_{*h}$. It is placed between the event and the ultimate horizon. It corresponds to the minimum value that the Rindler coordinate takes just before the disappearance of the black hole. It provides an upper bound for the redshift that an external observer can detect on an outgoing Hawking particle.
- 3) *Event horizon*: Defined as usual in general relativity as the boundary of the black hole.
- 4) *Apparent horizon*: Defined in the standard way as the boundary of the total trapped surface (see e.g. ref. [11] and appendix A).
- 5) *Stretched and global apparent horizons*: Let r_{*e} be the value of r_* at the endpoint of Hawking radiation (the endpoint is the intersection between the event horizon and the singularity line). We define the *stretched horizon* to be the time-like surface of space-time points with $r_* = r_{*e}$. The *global apparent horizon* is the time-like surface which coincides with the apparent horizon after the incoming energy flux has stopped and, for earlier times, it is defined by analytic continuation. For large black holes the stretched and the global

apparent horizons are the same object, as will be shown below. This horizon is of relevance in describing the evaporation of a black hole. In particular, the bulk of Hawking radiation originates at the global apparent horizon. As the black hole evaporates, the radius on this horizon recedes in correspondence with the mass loss of the black hole. The proper distance from these horizons to the event horizon, $\delta s = \int_{e.h.}^{s.h.} dr \sqrt{g_{rr}}$, and the local Unruh temperature, are of Planck magnitude (times a constant depending also on the number of propagating fields). The stretched (or global apparent) horizon coincides with the event horizon in the limit where l_{Planck} , or \hbar , or the number N of propagating local degrees of freedom goes to zero.

In the context of the information loss problem, the “stretched” horizon was discussed in ref. [10].¹ There it was described as a time-like surface where the local Unruh temperature is of Planck order, but the exact value was not specified. The above definition determines a very specific time-like surface. It is important here to determine the exact location of the global apparent horizon for the calculation of the Bekenstein entropy.

The present results may be interpreted as follows. Let us suppose we use the standard field-theoretic formula for entropy, and then ask where the brick wall must be placed in order to agree with the usual formula. For two-dimensional black holes the wall must be placed at an exponentially small distance from the ultimate horizon, with a particular N dependence. It turns out that the same dependence governs the separation between the ultimate horizon and the event horizon. It is thus natural to locate the entropy in the inaccessible region between these two horizons. In four dimensions, the same reasoning will locate the entropy on the stretched horizon. Since the bulk of the Hawking radiation is in the region which is in causal contact with the stretched horizon, and it is causally disconnected from the hidden horizon, the above suggests that in two dimensions the quantum mechanical information cannot reappear in the Hawking radiation, whereas in four dimensions Hawking particles can conceivably carry the information.

2. Entropy in two-dimensional gravity

The black hole entropy in two-dimensional dilaton gravity (as can be derived, e.g. from thermodynamics or as a surface term) is given by:

$$S = \frac{2M}{\lambda} \quad . \quad (2.1)$$

Different discussions of black hole entropy in two-dimensional dilaton gravity can be found e.g. in ref. [13]. A statistical mechanical (or fine-grained) explanation of the entropy (2.1)

¹ For the role of a *stretched horizon* in classical black hole physics, see ref. [12].

is unknown. To derive the black hole entropy from statistical mechanics, we will count field eigenmodes near the horizon following refs. [5] and [6].

Let us first briefly review the approach where the black hole is treated as a classical fixed background [6]. In Kruskal coordinates the classical black hole geometry formed by the collapse of a shock wave at time x_0^+ is given by

$$ds^2 = -\frac{dx^+ dx^-}{\frac{M}{\lambda} - \lambda^2 x^+ (x^- + p)} , \quad e^{-2\phi} = \frac{M}{\lambda} - \lambda^2 x^+ (x^- + p) , \quad p \equiv \frac{M}{\lambda^3 x_0^+} . \quad (2.2)$$

Rindler coordinates are introduced by

$$\frac{M}{\lambda} \sinh^2(\lambda R) = -\lambda^2 x^+ (x^- + p) , \quad e^{2\lambda T} = -\frac{x^+}{x^- + p} , \quad (2.3)$$

$$ds^2 = dR^2 - \tanh^2(\lambda R) dT^2 , \quad \phi = -\frac{1}{2} \log \left[\frac{M}{\lambda} \cosh^2(\lambda R) \right] . \quad (2.4)$$

Near the event horizon, $R \cong 0$, and the black hole metric is well approximated by the Rindler metric $ds^2 = dR^2 - \lambda^2 R^2 dT^2$. Consider N massless scalar fields f_i propagating in this background. In Kruskal coordinates the field equation is simply $\partial_+ \partial_- f_i = 0$. The solution in Rindler coordinates can be written as $f(R, T) = \varphi(R) e^{i\omega T}$, where

$$\varphi(R) = A \cos(\omega \sigma) + B \sin(\omega \sigma) , \quad \lambda \sigma = \log(\lambda R) .$$

In the spirit of the “brick wall” model, we will introduce an ultraviolet cutoff $R = \varepsilon$ and, in addition, an infrared cutoff at $R = L$. Assuming periodic boundary conditions, the eigenvalues for ω are then $\omega_n = \frac{\pi n}{L}$, where $l = \frac{1}{\lambda} \log(L/\varepsilon)$, $n = 1, 2, \dots$. The free energy is given by

$$F(\beta) = \frac{N}{\beta} \sum_{n=1}^{\infty} \log(1 - e^{-\beta \omega_n}) . \quad (2.5)$$

For large L ,

$$\sum_{n=1}^{\infty} \log(1 - e^{-na}) = - \sum_{m=1}^{\infty} \frac{e^{-ma}}{m(1 - e^{-ma})} \cong -\frac{\pi^2}{6} \frac{1}{a} , \quad a = \frac{\pi \lambda \beta}{\log(L/\varepsilon)} .$$

Thus

$$F(\beta) = -\frac{\pi N}{6 \lambda \beta^2} \log \frac{L}{\varepsilon} . \quad (2.6)$$

Using $S = \beta^2 \frac{\partial F}{\partial \beta}$ and the fact that $\beta = 2\pi/\lambda$, the entropy is found to be given by

$$S = \frac{N}{6} \log \frac{L}{\varepsilon} . \quad (2.7)$$

The entropy has an ultraviolet divergence as $\varepsilon \rightarrow 0$ due to an accumulation of modes near the horizon.

Now it will be shown that the result is actually finite upon inclusion of back-reaction effects. The evaporation makes that the event and the ultimate horizon split. The field-theoretical formula cannot be extrapolated up to the ultimate horizon, because the black hole has disappeared long before. As a result, physical quantities relevant for time-like external observers always remain finite, taking their maximum value near the event horizon. In particular, the maximum redshift that an outside observer can detect on an outgoing particle is given in terms of the value of g_{00} on the minimum value that the Rindler coordinate takes just before the disappearance of the black hole. The “hidden horizon” regularizes all quantities which would classically diverge if the evaporation is not included.

Let us consider an arbitrary distribution $T_{++}(x^+)$ of incoming matter with an energy density flux above the threshold for black hole formation [9]. The collapse starts at time x_0^+ and finishes at time x_1^+ . In Kruskal coordinates, the resulting time-dependent geometry is given by $ds^2 = -e^{2\rho}dx^+dx^-$, $\rho = \phi$ and

$$2\kappa\phi + e^{-2\phi} = -\lambda^2 x^+ (x^- + \lambda^{-2} P_+(x^+)) + \frac{1}{\lambda} M(x^+) - \kappa \log(-\lambda^2 x^+ x^-) , \quad (2.8)$$

$$\kappa = \frac{N}{48} , \quad P_+(x^+) = \int_{x_0^+}^{x^+} dx^+ T_{++}(x^+) , \quad M_+(x^+) = \lambda \int_{x_0^+}^{x^+} dx^+ x^+ T_{++}(x^+) .$$

The curvature singularity is at $\phi(x^+, x^-) = -\frac{1}{2} \log \kappa$. By definition the apparent horizon is at $\partial_+ \phi = 0$ [14] (see appendix A). This is the time-like curve

$$\lambda^2 x^+ (x^- + \lambda^{-2} P_+(x^+)) = -\kappa . \quad (2.9)$$

After the incoming flux has stopped, $x^+ > x_1^+$, we have $P_+(x^+) = P_+(x_1^+) \equiv \lambda^2 p$ and $M_+(x^+) = M_+(x_1^+) \equiv M$, and the equation of the apparent horizon becomes simply $\lambda^2 x^+ (x^- + p) = -\kappa$. The endpoint of the black hole, i.e. the point where the singularity becomes light-like, is at the intersection of the apparent horizon and the singularity line $\phi(x^+, x^-) = -\frac{1}{2} \log \kappa$. This occurs at

$$x_e^+ = \frac{\kappa}{\lambda^2 p} (e^{M/\kappa\lambda} - 1) , \quad x_e^- = -p(1 - e^{-M/\kappa\lambda})^{-1} . \quad (2.10)$$

The event horizon is thus the line $x^- = x_e^-$, $x^+ < x_e^+$. At $x^+ > x_e^+$, $x^- = x_e^-$ the geometry is matched with the Minkowski vacuum, $ds^2 = -d\tau^2 + d\sigma^2$, $\phi = -\lambda\sigma$, where

$$e^{2\lambda\sigma} = -\lambda^2 x^+ (x^- + p) , \quad e^{2\lambda\tau} = -\frac{x^+}{x^- + p} . \quad (2.11)$$

The ultimate horizon is at $\tau = \infty$, i.e. the null line $x^- = -p$. To determine the stretched horizon and the hidden horizon, we apply the definitions given in the previous section. Now r_* corresponds to σ . The stretched horizon is at $\sigma = \sigma_e = \frac{1}{2\lambda} \log[-\lambda^2 x_e^+(x_e^- + p)]$, which is the hyperbola

$$\lambda^2 x^+(x^- + p) = -\kappa . \quad (2.12)$$

As anticipated in section 1, this coincides with the apparent horizon for $x^+ > x_1^+$, and it is in this sense an analytic continuation of the apparent horizon in the region $x^+ < x_1^+$.

For large black holes the location of the global apparent horizon can also be determined without having an exact solution of the back-reaction problem. For $x^+ > x_1^+$ the equation $\partial_+ \phi = 0$ can be written as

$$0 = \frac{de^{-2\phi}}{dx^-} - \frac{\partial e^{-2\phi}}{\partial x^-} \cong \frac{de^{-2\phi}}{dx^-} + \lambda^2 x^+ . \quad (2.13)$$

Since the value of $e^{-2\phi}$ at the horizon is the mass of the black hole, $e^{-2\phi_h} = M/\lambda$, we can compute $\frac{de^{-2\phi}}{dx^-}$ from the mass loss rate. This can be obtained by differentiating the Bondi mass [9],

$$\frac{dM(x^-)}{dx^-} \cong \frac{\kappa\lambda}{x^- + p} .$$

Inserting this into eq. (2.13) the global apparent horizon is found at $\lambda^2 x^+(x^- + p) = -\kappa$, in exact agreement with eq. (2.12).

Similarly, using $x_0^+(x_e^- + p) = -px_0^+(e^{M/\kappa\lambda} - 1)^{-1}$, the hidden horizon is found at

$$x^+(x^- + p) = -px_0^+(e^{M/\kappa\lambda} - 1)^{-1} . \quad (2.14)$$

Now we would like to count all field configurations that are outside the black hole. The main contribution comes, as calculated above, from field configurations near the event horizon, where the geometry (for a large black hole) is very accurately described by the Rindler metric. The previous calculation applies, except that now the Rindler coordinate, defined by $MR^2 = -\lambda x^+(x^- + p)$, does not start at zero. The minimum value that the Rindler coordinate takes before the black hole disappears is at the intersection of the event horizon and the hidden horizon:

$$R_{\min}^2 = \frac{\lambda p x_0^+}{M} (e^{M/\kappa\lambda} - 1)^{-1} . \quad (2.15)$$

The curve $R = R_{\min}$ is the hidden horizon given by eq. (2.14). Thus we find that the fine-grained entropy associated with fields near the event horizon is

$$S = \frac{N}{12} \log \frac{L^2}{R_{\min}^2} = \frac{4M}{\lambda} + \frac{N}{12} \log \left(\frac{L^2 M}{\lambda p x_0^+} \right) + O(e^{-M/\kappa\lambda}) . \quad (2.16)$$

The first term represents the black hole entropy in 2d dilaton gravity. The second term could be regarded as an additional quantum correction, but here it is of no relevance.² The interesting point here is having obtained $S = \frac{4M}{\lambda}$ from just statistical mechanics and quantum field theory. It is finite, and it differs from eq. (2.1) by only a factor of 2 (this may be due to different reasons, we will not speculate on this here). Interestingly, the factor N has cancelled out and S is of order $1/\hbar$ once \hbar is restored in the formulas.

By separating right and left moving-mode contributions, eq. (2.7) can be written in the following way:

$$S = \frac{N\pi T}{6\lambda} \left[\log \frac{x_L^- + p}{x_\varepsilon^- + p} + \log \frac{x_L^+}{x_\varepsilon^+} \right]. \quad (2.17)$$

Let us consider the entropy density, $\mathcal{S} = \frac{\partial S}{\partial \sigma^-}$, $\sigma^- = \tau - \sigma$ (see eq. (2.11)). Using the thermodynamic relation, $Td\mathcal{S} = d\mathcal{E}$, the formula for the outgoing energy density is recovered: $\mathcal{E} = \frac{N\pi T^2}{12} = N \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega}{e^{\omega/T} - 1}$. For $x^- > x_e^-$ there is no more Hawking radiation, so in this region one has $\mathcal{E} = 0$. Thus the identification $x_\varepsilon^- = x_e^-$, that was made in deriving eq. (2.16), is nothing but the statement that the Hawking energy-density flux stops at $x^- = x_e^-$. The correct dependence on M has emerged in eq. (2.16), by just taking into account the physical fact that after some finite time the black hole disappears into the vacuum.

3. Entropy in four-dimensional gravity

The calculation of the entropy in the case of the Schwarzschild black hole using the brick wall model is similar to the calculation in the first part of the previous section. The Rindler coordinate is

$$R^2 = 8GM(r - 2GM). \quad (3.1)$$

The following expression is found in terms of a Rindler cutoff $R = \varepsilon$ [6] (see also ref. [5]):

$$S = \frac{NA}{360\pi\varepsilon^2}, \quad A = 16\pi G^2 M^2. \quad (3.2)$$

Let us now take into account the back reaction in the geometry. Although a full treatment including the evaporation cannot be performed by using exact analytic methods, for large black holes it is possible to make some very accurate estimates. The minimum value that the Rindler coordinate R can take outside the event horizon occurs at the point where the hidden horizon begins. This value is calculated in appendix B,

$$R_{\min}^2 = \varepsilon^2 = \frac{NG}{60\pi} e^{-bGM^2}, \quad b = \frac{320\pi}{N}. \quad (3.3)$$

² Note that for a shock wave collapse this term simplifies, $\log\left(\frac{L^2 M}{\lambda p x_0^+}\right) = 2\log(\lambda L)$.

This gives

$$S = \frac{A}{6G} e^{bGM^2} . \quad (3.4)$$

Equation (3.4) represents the entanglement entropy as predicted by quantum field theory; it is finite, but exponentially larger than the Bekenstein entropy, eq. (1.1). Thus, unlike the two-dimensional case, the straightforward calculation of the entropy that takes into account the evaporation does not give a reasonable result. This can be viewed as a failure of conventional quantum field theory to correctly describe physics in the vicinity of the event horizon [5]. A short-distance cutoff is needed before.

From the point of view of the information problem, this is actually fortunate: if the entropy was to be associated with degrees of freedom near the hidden horizon (or near any region at an exponentially small distance from the event horizon), this would strongly suggest that the information could not possibly come out with the Hawking radiation. Hawking radiation originates in the region causally connected with the global apparent horizon. Let us thus postulate –following [5,10]– that the quantum mechanical information reappears in the Hawking radiation. The natural place for the wall in this scenario is the global apparent horizon. This is a very specific surface, so this ansatz will provide an unambiguous prediction for the Bekenstein entropy. If this prediction does not agree with the expression for the black hole entropy, eq. (1.1), then this will mean that this ansatz cannot be correct. Surprisingly, we will find an almost exact agreement.

Let the final Minkowski metric be given by

$$ds^2 = -dt^2 + dr_*^2 + r_*^2 d\Omega^2 . \quad (3.5)$$

Let us introduce standard Kruskal coordinates

$$-\frac{1}{2MG} V(U + 2MG) = 2MG e^{r_*/2MG} , \quad -\frac{V}{U + 2MG} = e^{t/2MG} . \quad (3.6)$$

or (cf. eq. (2.11))

$$V = 2MG e^{v/4MG} , \quad U + 2MG = -2MG e^{-u/4MG} , \quad v, u = t \pm r_* . \quad (3.7)$$

After the incoming flux of collapsing matter stops, apparent and global apparent horizons coincide. For spherically symmetric configurations the apparent horizon is at $\frac{\partial r(U,V)}{\partial V} = 0$ (see appendix A). For a large black hole (away from the endpoint) the geometry is given by the Schwarzschild metric with

$$2MG(r - 2MG) e^{\frac{r}{2MG}} = -V(U + 2MG) . \quad (3.8)$$

Near the horizon, $r \cong 2MG$, and we can solve this transcendental equation for r iteratively. We are only interested in the leading part, which is

$$r \cong -\frac{1}{2eMG}V(U + 2MG) + 2MG + O\left(\frac{M_{\text{Planck}}^2}{M^3}\right). \quad (3.9)$$

We will determine the global apparent horizon by solving the equation $\frac{\partial r(U,V)}{\partial V} = 0$ after the incoming flux of collapsing matter has stopped, just as we did in the previous section (see eq. (2.13)). This equation can be written as

$$0 = \frac{dr_{\text{GAH}}}{dU} - \frac{\partial r_{\text{GAH}}}{\partial U} \cong \frac{dr_{\text{GAH}}}{dU} + \frac{1}{2eMG}V. \quad (3.10)$$

For a large Schwarzschild black hole, $r_{\text{GAH}} \cong 2MG$, so that

$$\frac{dr_{\text{GAH}}}{dU} \cong 2G \frac{dM}{dU}. \quad (3.11)$$

Now, in the vicinity of the horizon, a black hole loses mass at a rate

$$\frac{dM}{du} = -N \frac{\pi^2}{15} T^4 A = -\frac{N}{3840\pi G^2} \frac{1}{M^2}. \quad (3.12)$$

Thus

$$\frac{dM}{dU} = \frac{N}{960\pi G(U + 2MG)} \frac{1}{M}. \quad (3.13)$$

Using eqs. (3.10), (3.11) and (3.13), we find the following equation for the (global) apparent horizon (cf. eq. (2.12)) :

$$V(U + 2MG) \cong -\frac{NeG}{240\pi}. \quad (3.14)$$

It is clear that for a large black hole (i.e. $M \gg M_{\text{Planck}}$) the global apparent horizon coincides with the stretched horizon. Indeed, the apparent horizon must pass through the black hole endpoint (from the definition of “endpoint”). Comparing with eq. (3.6), we see that the equation of the global apparent horizon (3.14) (which is correct up to terms of $O(M_{\text{Planck}}^2/M^2)$) can be simply written as $r_* = r_{*e}$. This is precisely our definition of stretched horizon.

In terms of r this is the time-like surface $r = r_{\text{S.H.}} = 2MG + \delta$, with (see eq. (3.8))

$$r_{\text{S.H.}} \cong 2MG + \frac{N}{480\pi} \frac{1}{M}. \quad (3.15)$$

Let us check that the proper distance ds from this surface to the event horizon is indeed of Planck order. We have ($\delta \ll 2MG$):

$$ds = \int_{2MG}^{2MG+\delta} \frac{dr}{\sqrt{1 - \frac{2MG}{r}}} \cong 2\sqrt{2MG\delta} = \frac{1}{2}\sqrt{\frac{NG}{15\pi}}. \quad (3.16)$$

We see that the proper distance increases with the square root of the number of scalar fields. Using the expression for the Rindler coordinate, eq. (3.1), we find $R_{\text{S.H.}}^2 = \frac{NG}{60\pi}$, whence we finally obtain

$$S = \frac{2}{3} \frac{A}{4G\hbar} , \quad (3.17)$$

which differs from the Bekenstein entropy by only a factor of $2/3$.

To summarize, in two dimensions the standard field-theoretic formula for the entropy, excluding contributions beyond R_{min} , reproduces the usual expression for the black hole entropy up to a numerical constant of order 1. In four dimensions, quantum field theory predicts too large a fine-grained entropy given by eq. (3.4). An ultraviolet cutoff is necessary before the event horizon, perhaps due to a breakdown of conventional quantum field theory [5]. For the information to come out in the Hawking radiation, it is natural to locate the entropy on the global apparent horizon, since this is precisely the place where most Hawking particles originate. The location of this horizon was calculated and, strikingly, the Bekenstein entropy formula has emerged, independent of the number of propagating fields, with the correct dependence on the mass, the Newton constant and \hbar . These results may be viewed as an indication that in four dimensions the quantum mechanical information is encoded in long-time correlations in the Hawking radiation [5,10,15].

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Appendix A. Apparent horizon for spherically symmetric configurations

In this appendix we discuss the connection between the two-dimensional definition of apparent horizon, $\partial_+\phi\partial_-\phi = 0$, and the general four-dimensional definition (a further discussion on the latter can be found, e.g. in ref. [11]).

Let C be a three-dimensional manifold with boundary S . Let ξ_μ , $\mu = 0, 1, 2, 3$, be the vector field of tangents to a congruence of outgoing null geodesics orthogonal to S . C is a *trapped region* if the expansion $\theta = \nabla_\mu \xi^\mu$ is everywhere non-positive on S , $\theta \leq 0$. The *apparent horizon* \mathcal{A} is the boundary of the *total trapped region*, the latter defined as the closure of the union of all trapped regions. A corollary of this definition is that $\theta = 0$ on \mathcal{A} .

Let us now contemplate metrics of the form

$$ds^2 = g_{ij}(x^0, x^1)dx^i dx^j + \exp[-2\phi(x^0, x^1)]d\Omega^2 , \quad i, j = 0, 1 . \quad (\text{A.1})$$

In this spherically symmetric space-time, we have $\xi_\mu = \{\xi_0, \xi_1, 0, 0\}$, and the geodesic equation reduces to

$$\xi^i \nabla_i \xi^j = 0 , \quad (\text{A.2})$$

i.e. the two-dimensional geodesic equation. Since in this dimensionally reduced configuration there is only one family of outgoing null geodesics, a trapped region is the total trapped region, and the condition determining the apparent horizon simply becomes

$$\theta = 0 . \quad (\text{A.3})$$

From eq. (A.1) one easily obtains

$$\theta = \theta^{(2)} - 2\xi^j \partial_j \phi , \quad (\text{A.4})$$

where $\theta^{(2)} \equiv \partial_i \xi^i + \Gamma_{ij}^i \xi^j$. Let us denote $B_{ik} = \nabla_i \xi_k$. By using the geodesic equation, $\xi^i B_{ik} = 0$, and from the fact that ξ is null, $\xi^i \xi_i = 0$, the following relations can be derived:

$$\xi^1 B_{11} = -\xi^2 B_{21} , \quad \xi^1 B_{21} = -\xi^2 B_{22} , \quad B_{12} = B_{21} , \quad (\text{A.5})$$

from which we obtain

$$\begin{aligned} \theta^{(2)} &= g^{ij} B_{ij} = B_{11} [g^{11} - 2 \frac{\xi^1}{\xi^2} g^{12} + (\frac{\xi^1}{\xi^2})^2 g^{22}] \\ &= 0 , \end{aligned} \quad (\text{A.6})$$

where we have used $\xi^1 \xi_1 = -\xi^2 \xi_2$. We thus see that the two-dimensional expansion parameter is identically zero (in particular, this means that an intrinsically two-dimensional apparent horizon cannot be defined). Now, by using eqs. (A.3), (A.4) and (A.6), we find that the condition defining the apparent horizon becomes

$$\xi^i \partial_i \phi = 0 . \quad (\text{A.7})$$

Since ξ is null, eq. (A.7) implies $\xi_i = f(x) \partial_i \phi$, where $f(x)$ is a scalar function. Therefore the condition (A.3) translates to

$$g^{ij} \partial_i \phi \partial_j \phi = 0 , \quad (\text{A.8})$$

or, in the conformal gauge, $\partial_+ \phi \partial_- \phi = 0$, which thus determines the location of the apparent horizon \mathcal{A} .

Appendix B. Hidden horizon in four dimensions

Here we calculate the value of the Kruskal coordinates at the endpoint of an evaporating four-dimensional black hole, as well as the location of the hidden horizon. Let us assume that the black hole is formed by the collapse of a shock wave at advanced time $v = v_0 = 0$ (so that $V_0 = 2MG$). The black hole will Hawking-radiate at a rate

$$\frac{dM(v)}{dv} = -\frac{N}{3840\pi G^2} \frac{1}{M^2(v)} , \quad (\text{B.1})$$

i.e.

$$\frac{N}{1280\pi G^2}v = M^3 - M^3(v) . \quad (\text{B.2})$$

Thus (see eq. (3.7))

$$V_e = 2MG e^{bGM^2} , \quad b = \frac{320\pi}{N} . \quad (\text{B.3})$$

Since the apparent horizon passes through the endpoint, we can use eq. (3.14) to determine U_e :

$$U_e + 2GM = -\frac{Ne}{480\pi M} e^{-bGM^2} . \quad (\text{B.4})$$

By definition the hidden horizon is at $r = r_{*h}$, where

$$2MG e^{r_{*h}/2MG} = -\frac{1}{2MG} V_0(U_e + 2MG) . \quad (\text{B.5})$$

This is the hyperbola

$$V(U + 2MG) = -\frac{NeG}{240\pi} e^{-bGM^2} . \quad (\text{B.6})$$

The Rindler coordinate is $R^2 = 8MG(r - 2MG) \cong -4e^{-1}V(U + 2MG)$. This gives eq. (3.3).

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